Lie Group S-Expansions and Infinite-dimensional Lie algebras

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Abstract

The expansion method of Lie algebras by a semigroup or S-expansion is generalized to act directly on the group manifold, and not only at the level of its Lie algebra. The consistency of this generalization with the dual formulation of the S-expansion is also verified. This is used to show that the Lie algebras of smooth mappings of some manifold M onto a finite-dimensional Lie algebra, such as the so called loop algebras, can be interpreted as a particular kind of S-expanded Lie algebras. We consider as an example the construction of a Yang-Mills theory for an infinite-dimensional algebra, namely loop algebra.

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I. INTRODUCTION

Contractions of finite dimensional Lie algebras were introduced several decades ago by Inönü and Wigner and successfully applied to recover the Galilei group from the Lorentz group. Subsequently, group contractions were used to retrieve the Poincaré group from the de-Sitter group in various dimensions.

In Ref. [1] were discussed the Inönü-Wigner contractions of affine Kac-Moody algebras and in Ref. [2] were studied contractions of some infinite-dimensional Lie algebras such as Kac-Moody, Witt, Virasoro and Krichever-Novikov algebras

Expansions of finite dimensional Lie algebras were introduced in Refs. [3], [4], [5], [6], and successfully applied to recover the M-algebra as well as the so called D'Auria-Fré-algebra from the $OSp(32/1)$ Lie algebra. Subsequently, Lie algebras expansions were used to construct invariant tensor, Chern-Simons and Transgression forms for the expanded algebras [7], [8], [9].

Recently in Refs. [10], [11], the expansion methods were generalized to obtain expanded higher-order Lie algebras and in Ref. [12] was constructed the dual formulation of the Lie algebra S-expansion procedure whose application permits, for example, to obtain the $(2+1)$-dimensional Chern-Simons AdS gravity from the so-called "exotic gravity".

It is the purpose of this paper to show that the Lie algebras of smooth mappings of some manifold M onto a finite-dimensional Lie algebra, such as the so called loop algebras, can be interpreted as a particular kind of S-expanded Lie algebras, where the notion of S-expansion is generalized to act directly on the group manifold and not only at the level of the Lie algebra. We consider as an example the construction of a Yang-Mills theory for the loop algebra.

II. LIE GROUP EXPANSIONS

A. The S-expansion procedure

In this section we shall review some aspects of the S-expansion procedure introduced in ref. [6], [10]. The S-expansion method is based on combining the structure constants of the Lie algebra $(g, [,])$ with the inner law of a semigroup S to define the Lie bracket of a new, S-expanded algebra. Let $S = \{\lambda_\alpha\}$ be a finite Abelian semigroup endowed with a commutative

and associative composition law $S\times S\rightarrow S$, $(\lambda_\alpha,\lambda_\beta)\mapsto\lambda_\alpha\lambda_\beta=K_{\alpha\beta}{}^{\gamma}\lambda_\gamma$. Let the pair $(g,[,])$ be a Lie algebra where g is a finite dimensional vector space, with basis $\{\mathbf{T}_A\}_{A=1}^{\dim g}$, over the field K; and $[,]$ is a ruler of composition $g\times g\longrightarrow g$, $(\mathbf{T}_A,\mathbf{T}_B)\longrightarrow[\mathbf{T}_A,\mathbf{T}_B]=C_{AB}^{\ \ C}\mathbf{T}_C$. The direct product $\mathcal{G}=S\otimes g$ is defined as the Cartesian product set

$$\mathcal{G}=S\otimes g=\left\{\mathbf{T}_{(A,\alpha)}=\lambda_\alpha\mathbf{T}_A:\ \lambda_\alpha\in S\ ,\ \mathbf{T}_A\in g\right\}, \tag{1}$$

endowed with a composition law $[,]_S:\mathcal{G}\times\mathcal{G}\rightarrow\mathcal{G}$ defined by

$$\left[\mathbf{T}_{(A,\alpha)},\mathbf{T}_{(B,\beta)}\right]_S=:\lambda_\alpha\lambda_\beta\left[\mathbf{T}_A,\mathbf{T}_B\right]=K_{\alpha\beta}{}^{\gamma}C_{AB}^{\ \ C}\lambda_\gamma\mathbf{T}_C=C_{(A,\alpha)(B,\beta)}^{\qquad\quad (C,\gamma)}\mathbf{T}_{(C,\gamma)}, \tag{2}$$

where $\mathbf{T}_{(A,\alpha)}=\lambda_\alpha\mathbf{T}_A$ is a basis of $\mathcal{G}$. The set (1) with the composition law (2) is called a S-expanded Lie algebra. This algebra is a Lie algebra structure defined over the vector space obtained by taking S copies of g

$$\mathcal{G}\colon\oplus_{\alpha\in S}W_\alpha\ \ (W_\alpha\approx g,\,\forall\alpha)\,, \tag{3}$$

$\dim\mathcal{G}=ordS\times\dim g$ by means of the structure constants

$$C_{(A,\alpha)(B,\beta)}^{\qquad\quad (C,\gamma)}=C_{AB}^{\ \ C}\delta_{\alpha\beta}^{\gamma}, \tag{4}$$

where δ is the Kronecker symbol and the subindex $\alpha,\beta\in S$ denotes the inner composition in S so that $\delta_{\alpha\beta}^{\gamma}=1$ when $\alpha\beta=\gamma$ in S and zero otherwise. The constants $C_{(A,\alpha)(B,\beta)}^{\qquad\quad (C,\gamma)}$ defined by (4) inherit the symmetry properties of $C_{AB}^{\ \ C}$ of g by virtue of the abelian character of the S-product, and satisfy the Jacobi identity.

In a nutshell, the S-expansion method can be seen as the natural generalization of the Inönü-Wigner contraction, where instead of multiplying the generators by a numerical parameter, we multiply the generators by the elements of an Abelian semigroup.

Theorem II.1. *The product $[,]_S$ defined in (2) is also a Lie product because it is linear, antisymmetric and satisfies the Jacobi identity. This product defines a new Lie algebra characterized by the pair $(\mathcal{G},[,]_S)$, and is called a S-expanded Lie algebra.*

Proof. The proof is direct and may be found in [6], [10]. □

B. S-expansion of Lie groups

We can reinterpret the S-expansion procedure as a method to obtain a new Lie group $\widetilde{G}$ from a given original Lie group G, i.e. not only as a relation between two different Lie algebras but as a method that relates two different Lie groups[14].

Definition II.1. *Let γ be an element of the original Lie group G parametrized as $\gamma = \exp\left[\theta^A \mathbf{T}_A\right]$, where $\left\{\theta^A\right\}$, $A = 1, .., \dim G$ are the group coordinates and $\left\{\mathbf{T}_A\right\}$ are the generators of associated Lie algebra g with commutation relations $\left[\mathbf{T}_A, \mathbf{T}_B\right] = C_{AB}^{\ \ C}\mathbf{T}_C$, and let $S = \{\lambda_\alpha\}_{\alpha=1}^{N+1}$ be an abelian semigroup.*

Definition II.2. *The* ***S-mapping*** *is the mapping from the semigroup S to the Lie group G , $\lambda_\alpha \longmapsto \gamma(\lambda_\alpha) \in G$ defined by*

$$\theta^A(\lambda) = \sum_{\alpha=1}^{N+1} \theta^{(A,\alpha)}\lambda_\alpha, \tag{5}$$

and the parameters $\left\{\theta^{(A,\alpha)}\right\}$ are called the ***S-mapping parameters.***

The S-mapping is completely characterized by the parameters $\left\{\theta^{(A,\alpha)}\right\}$ which, as we will show in the following theorem, are the group coordinates of the expanded Lie group $\tilde{G}$.

Theorem II.2. *The S-mapping parameters $\left\{\theta^{(A,\alpha)}\right\}$ define the coordinates of a new Lie group $\tilde{G}$, called the S-expanded Lie group, whose associated Lie algebra is the S-expanded Lie algebra $\mathcal{G} = S \otimes g$. The elements of the S-expanded Lie group are given by $\gamma(\lambda) = \exp\left[\theta^A(\gamma)\mathbf{T}_A\right]$.*

Proof. Replacing the expression for the S-mapping (5) in the parametrization of a group element we obtain

$$\begin{aligned}\gamma(\lambda) &= \exp\left[\theta^A(\gamma)\mathbf{T}_A\right], \\ &= \exp\left[\sum_\alpha \theta^{(A,\alpha)}\lambda_\alpha \mathbf{T}_A\right],\end{aligned} \tag{6}$$

by defining $\mathbf{T}_{(A,\alpha)} \equiv \lambda_\alpha \mathbf{T}_A$ we have

$$\gamma(\lambda) = \exp\left[\sum_\alpha \theta^{(A,\alpha)}\mathbf{T}_{(A,\alpha)}\right]. \tag{7}$$

The commutation relations of the $\left\{\mathbf{T}_{(A,\alpha)}\right\}$ are given by

$$\left[\mathbf{T}_{(A,\alpha)}, \mathbf{T}_{(B,\beta)}\right] = \left[\lambda_\alpha \mathbf{T}_A, \lambda_\beta \mathbf{T}_B\right] = \lambda_\alpha\lambda_\beta\left[\mathbf{T}_A, \mathbf{T}_B\right] = \lambda_\alpha\lambda_\beta\left(C_{AB}^{\ \ C}\mathbf{T}_C\right), \tag{8}$$

since

$$\lambda_\alpha\lambda_\beta = K_{\alpha\beta}^{\ \ \gamma}\lambda_\gamma, \tag{9}$$

we have

$$\left[\mathbf{T}_{(A,\alpha)},\mathbf{T}_{(B,\beta)}\right]=K_{\alpha\beta}{}^{\gamma}\lambda_{\gamma}\left(C_{AB}{}^{C}\mathbf{T}_{C}\right)=K_{\alpha\beta}{}^{\gamma}\left(C_{AB}{}^{C}\lambda_{\gamma}\mathbf{T}_{C}\right), \tag{10}$$

$$\left[\mathbf{T}_{(A,\alpha)},\mathbf{T}_{(B,\beta)}\right]=K_{\alpha\beta}{}^{\gamma}C_{AB}{}^{C}\mathbf{T}_{(C,\gamma)}. \tag{11}$$

The generators $\mathbf{T}_{(A,\alpha)}$ obey the commutation relations of the S-expanded algebra $\mathcal{G}=S\otimes g$, and according to the eq.(7) the S-mapping parameters $\left\{\theta^{(A,\alpha)}\right\}$ are the group coordinates of the S-expanded group $\tilde{G}$ with Lie algebra $\mathcal{G}=S\otimes g$. This completes the proof. $\square$

This theorem establishes the relation between the original Lie group G with coordinates $\left\{\theta^{A}\right\}$ and the S-expanded Lie group $\tilde{G}$ with coordinates $\left\{\theta^{(A,\alpha)}\right\}$ through the eq.(5), where the corresponding Lie algebras are related through an S-expansion as described in ref. [6]. In this context the expansion procedure occurs at the level of the group coordinates.

C. Consistency with dual formulation of the S-expansion procedure

In the previous section we have extended the notion of a S-expansion to the group manifold, where the expansion is done at the level of the group coordinates. As the Maurer-Cartan forms of the Lie group can be realized as left-invariant one-forms on the group manifold, the S-expansion at the group coordinates level should lead automatically to an S-expansion formulation in terms of the Maurer-Cartan forms. In the next theorem we will prove that this expansion over the left-invariant forms is consistent with the dual formulation of the S-expansion procedure of ref. [12].

Theorem II.3. *Let $\left\{\omega^{A}\left(\theta^{B}\right)\right\}$ be the Maurer-Cartan forms of the Lie group G and let $\left\{\omega^{(A,\alpha)}\left(\theta^{(B,\beta)}\right)\right\}$ be the Maurer-Cartan forms of the expanded Lie group $\tilde{G}$ realized as left invariant one-forms over the corresponding group manifold. The application of the S-expansion procedure on the group coordinates through an S-mapping (5) gives the following relation between the Maurer-Cartan forms of G and $\tilde{G}$:*

$$\omega^{A}=\sum_{\alpha=1}^{N+1}\lambda_{\alpha}\omega^{(A,\alpha)}, \tag{12}$$

expression consistent with the dual formulation of the S-expansion procedure of ref. [12].

Proof. The canonical one form left-invariant on G is given by

$$\gamma^{-1}d\gamma=e^{-\theta^{A}\mathbf{T}_{A}}de^{\theta^{A}\mathbf{T}_{A}}\equiv\omega^{A}\mathbf{T}_{A}. \tag{13}$$

Using the identity

$$e^{-\mathbf{X}}de^{\mathbf{X}}=d\mathbf{X}+\frac{1}{2}\left[d\mathbf{X},\mathbf{X}\right]+\frac{1}{3!}\left[\left[d\mathbf{X},\mathbf{X}\right],\mathbf{X}\right]+\frac{1}{4!}\left[\left[\left[d\mathbf{X},\mathbf{X}\right],\mathbf{X}\right],\mathbf{X}\right]+...,\tag{14}$$

we can obtain the left invariant Maurer-Cartan forms on G in terms of the group coordinates

$$\begin{aligned}\omega^{A}\left(\theta\right)&=d\theta^{A}+\frac{1}{2}C_{BE_{1}}{}^{A}\theta^{E_{1}}d\theta^{B}+\frac{1}{3!}C_{BE_{1}}{}^{D_{1}}C_{D_{1}E_{2}}^{A}\theta^{E_{1}}\theta^{E_{2}}d\theta^{B}\\&+\frac{1}{4!}C_{BE_{1}}{}^{D_{1}}C_{D_{1}E_{2}}{}^{D_{2}}C_{D_{2}E_{3}}{}^{A}\theta^{E_{2}}\theta^{E_{2}}\theta^{E_{3}}d\theta^{B}+...\end{aligned}\tag{15}$$

Introducing eq.(5) into this expression we have

$$\begin{aligned}\omega^{A}=&\left(\sum_{\alpha}\lambda_{\alpha}d\theta^{(A,\alpha)}\right)+\frac{1}{2}C_{BE_{1}}{}^{A}\left(\sum_{\varepsilon_{1}}\lambda_{\varepsilon_{1}}\theta^{(E_{1},\varepsilon_{1})}\right)\left(\sum_{\beta}\lambda_{\beta}d\theta^{(B,\beta)}\right)+\\&+\frac{1}{3!}C_{BE_{1}}{}^{D_{1}}C_{D_{1}E_{2}}{}^{A}\left(\sum_{\varepsilon_{1}}\lambda_{\varepsilon_{1}}\theta^{(E_{1},\varepsilon_{1})}\right)\left(\sum_{\varepsilon_{2}}\lambda_{\varepsilon_{2}}\theta^{(E_{2},\varepsilon_{2})}\right)\left(\sum_{\beta}\lambda_{\beta}d\theta^{(B,\beta)}\right)+\\&+\frac{1}{4!}C_{BE_{1}}{}^{D_{1}}C_{D_{1}E_{2}}{}^{D_{2}}C_{D_{2}E_{3}}{}^{A}\left(\sum_{\varepsilon_{1}}\lambda_{\varepsilon_{1}}\theta^{(E_{1},\varepsilon_{1})}\right)\left(\sum_{\varepsilon_{2}}\lambda_{\varepsilon_{2}}\theta^{(E_{2},\varepsilon_{2})}\right)\left(\sum_{\varepsilon_{3}}\lambda_{\varepsilon_{3}}\theta^{(E_{3},\varepsilon_{3})}\right)\left(\sum_{\beta}\lambda_{\beta}d\theta^{(B,\beta)}\right)+...,\end{aligned}\tag{16}$$

rearranging the semigroup elements

$$\begin{aligned}\omega^{A}=&\sum_{\alpha}\lambda_{\alpha}d\theta^{(A,\alpha)}+\frac{1}{2}C_{BE_{1}}{}^{A}\sum_{\varepsilon_{1},\beta}\lambda_{\varepsilon_{1}}\lambda_{\beta}\theta^{(E_{1},\varepsilon_{1})}d\theta^{(B,\beta)}+\frac{1}{3!}C_{BE_{1}}{}^{D_{1}}C_{D_{1}E_{2}}{}^{A}\sum_{\varepsilon_{1},\varepsilon_{2},\beta}\lambda_{\varepsilon_{1}}\lambda_{\varepsilon_{2}}\lambda_{\beta}\theta^{(E_{1},\varepsilon_{1})}\theta^{(E_{2},\varepsilon_{2})}d\theta^{(B,\beta)}\\&+\frac{1}{4!}C_{BE_{1}}{}^{D_{1}}C_{D_{1}E_{2}}{}^{D_{2}}C_{D_{2}E_{3}}{}^{A}\sum_{\varepsilon_{1},\varepsilon_{2},\varepsilon_{3},\varepsilon_{4}}\lambda_{\varepsilon_{1}}\lambda_{\varepsilon_{2}}\lambda_{\varepsilon_{3}}\lambda_{\beta}\theta^{(E_{1},\varepsilon_{1})}\theta^{(E_{2},\varepsilon_{2})}\theta^{(E_{3},\varepsilon_{3})}d\theta^{(B,\beta)}+.... \quad .\end{aligned}\tag{17}$$

Using the composition law of the semigroup $\lambda_{\alpha}\lambda_{\beta}=K_{\alpha\beta}{}^{\gamma}\lambda_{\gamma}$ and taken into account the fact that the two-selectors behave like a Kronecker delta, we can introduce a sum into α without changing the result:

$$\begin{aligned}\omega^{A}=&\sum_{\alpha}\lambda_{\alpha}dg^{(A,\alpha)}+\sum_{\alpha}\lambda_{\alpha}\frac{1}{2}\sum_{\varepsilon_{1},\beta}\left(C_{BE_{1}}{}^{A}K_{\beta\varepsilon_{1}}{}^{\alpha}\right)\theta^{(E_{1},\varepsilon_{1})}d\theta^{(B,\beta)}+\\&+\sum_{\alpha}\lambda_{\alpha}\frac{1}{3!}\sum_{\beta,\varepsilon_{1},\delta_{1},\varepsilon_{2}}\left(C_{BE_{1}}{}^{D_{1}}K_{\beta\varepsilon_{1}}{}^{\delta}\right)\left(C_{D_{1}E_{2}}{}^{A}K_{\delta\varepsilon_{2}}{}^{\alpha}\right)\theta^{(E_{1},\varepsilon_{1})}\theta^{(E_{2},\varepsilon_{2})}d\theta^{(B,\beta)}+\\&+\sum_{\alpha}\lambda_{\alpha}\frac{1}{4!}\sum_{\varepsilon_{1},\varepsilon_{2},\varepsilon_{3},\varepsilon_{4},\beta,\delta_{1},\delta_{2}}\left(C_{BE_{1}}{}^{D_{1}}K_{\beta\varepsilon_{1}}{}^{\delta_{1}}\right)\left(C_{D_{1}E_{2}}{}^{D_{2}}K_{\delta_{1}\varepsilon_{2}}{}^{\delta_{2}}\right)\left(C_{D_{2}E_{3}}{}^{A}K_{\delta_{2}\varepsilon_{3}}{}^{\alpha}\right)\theta^{(E_{1},\varepsilon_{1})}\theta^{(E_{2},\varepsilon_{2})}\theta^{(E_{3},\varepsilon_{3})}d\theta^{(B,\beta)}+..\end{aligned}\tag{18}$$

Introducing the structure constants of the expanded Lie algebra $C_{(A,\alpha)(B,\beta)}{}^{(C,\gamma)} := C_{AB}{}^{C} K_{\alpha\beta}{}^{\gamma}$, we have

$$\omega^A = \sum_{\alpha} \lambda_\alpha [\sum_{\beta} \delta_B^A \delta_\beta^\alpha + \frac{1}{2} \sum_{\varepsilon_1,\beta} C_{(B,\beta)(E_1,\varepsilon_1)}{}^{(A,\alpha)} \theta^{(E_1,\varepsilon_1)} + \frac{1}{3!} \sum_{\beta,\varepsilon_1,\delta_1.\varepsilon_2} C_{(B,\beta)(E_1,\varepsilon_1)}{}^{(D_1,\delta_1)} C_{(D_1,\delta_1)(E_2,\varepsilon_2)}{}^{(A,\alpha)} \theta^{(E_1,\varepsilon_1)} \theta^{(E_2,\varepsilon_2)}$$
$$+ \frac{1}{4!} \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\beta,\delta_1,\delta_2} C_{(B,\beta)(E_1,\varepsilon_1)}{}^{(D_1,\delta_1)} C_{(D_1,\delta_1)(E_2,\varepsilon_2)}{}^{(D_2,\delta_2)} C_{(D_2,\delta_2)(E_3,\varepsilon_3)}{}^{(A,\alpha)} \theta^{(E_1,\varepsilon_1)} \theta^{(E_2,\varepsilon_2)} \theta^{(E_3,\varepsilon_3)} + ...] d\theta^{(B,\beta)}. \tag{19}$$

Finally we can write

$$\omega^A = \sum_{\alpha} \lambda_\alpha \omega^{(A,\alpha)}, \tag{20}$$

where $\omega^{(A,\alpha)}$ are the Maurer-Cartan forms of the S-expanded algebra. This completes the proof. $\square$

The eq.(20) gives the relation between the Maurer-Cartan forms of the original and the expanded Lie algebras. This expression is in agreement with the dual formulation of the S-expansion method of ref. [12].

III. INFINITE-DIMENSIONAL LIE ALGEBRAS

In this section we shall review some aspects of the Lie algebras of smooth mapping of a manifold M onto a finite-dimensional Lie algebra. The main point of this section is to display the differences between the usual Lie algebras and the algebras for the 1-sphere, 2-sphere and 3-sphere groups of a compact simple Lie group.

A. The Affine Kac-Moody Algebra

Let g be the Lie algebra of a finite-dimensional compact simple Lie group, G. Let $Map(S^1, G)$ be the set of smooth mappings from the circle S^1 to G:

$$\gamma : S^1 \longmapsto G; \quad z \longmapsto \gamma(z), \tag{21}$$

where $z \in \mathbb{C}$, $|z| = 1$. Here $\mathbb{C}$ is the field of complex numbers. The set of $Map(S^1, G)$ acquires the structure of a group under the pointwise multiplication (in G) :

$$(\gamma_1 \circ \gamma_2)(z) = \gamma_1(z)\gamma_2(z). \tag{22}$$

This is the loop group of G. It has the natural structure of an infinite-dimensional Lie group.

To obtain the corresponding Lie algebra [13], denoted by $Map(S^1, G)$, we begin by choosing a basis $\mathbf{T}^{\,A}, 1 \leq A \leq \dim(g)$, for g with

$$\left[\mathbf{T}^A, \mathbf{T}^B\right] = i f^{ABC} \mathbf{T}^C, \tag{23}$$

where the structure constants f^{abc} satisfy the Jacobi identity

$$f^{ABD} f^{DCE} + f^{CAD} f^{DBE} + f^{BCD} f^{DAE} = 0, \tag{24}$$

and antisymmetry

$$f^{ABC} = -f^{BAC}. \tag{25}$$

Moreover, we choose f^{ABC} to be totally antisymmetric, satisfying

$$f^{ABC} f^{ABD} = \delta^{CD}, \tag{26}$$

which can always be done since G is restricted to be simple. The connected component of the group $Map(S^1, G)$ consists of elements

$$\gamma(z) = \exp\left(-i \mathbf{T}^A \theta_A(z)\right). \tag{27}$$

Near the identity they have the form

$$\gamma(z) \approx 1 - i \mathbf{T}^A \theta_A(z), \tag{28}$$

where the functions $\theta_A(z)$ are defined on the unit circle. They can be expanded in a Laurent series

$$\theta_A(z) = \sum_{n=-\infty}^{\infty} \theta_A^n z^n. \tag{29}$$

Thus

$$\gamma(z) \approx 1 - i \sum_{n=-\infty}^{\infty} \theta_A^n \mathbf{T}_n^{\,A}. \tag{30}$$

Identifying θ_A^n as the infinitesimal parameters of $Map(S^1, G)$ we see that $\mathbf{T}_n^A \equiv \mathbf{T}^{\,A} z^n$ are the generators of the algebra $Map(S^1, g)$. Note that (29) may be viewed as the expansion of $\theta_A(z)$ in a Fourier series; that is, in a complete set of functions on the unit circle. The basic commutation relations for the generators $\mathbf{T}_n^A$ may now be derived

$$\left[\mathbf{T}_m^A, \mathbf{T}_n^B\right] = i f^{ABC} \mathbf{T}_{m+n}^C. \tag{31}$$

On a basis where the $\mathbf{T}^A$ are hermitian operators, a consistent definition of the hermitian adjoint turns out to be

$$\mathbf{T}_n^{A\dagger} = \mathbf{T}_{-n}^A. \tag{32}$$

The loop algebra admits a nontrivial central extension. This central extension is unidimensional and essentialy unique. This is called the Kac-Moody algebra. The commutation relations of the Kac-Moody algebra (on the chosen basis) are

$$\left[\mathbf{T}_m^A, \mathbf{T}_n^B\right] = if^{ABC}\mathbf{T}_{m+n}^C + \mathbf{k}\delta^{AB} m\delta_{m,-n}, \tag{33}$$

where $\mathbf{k}$ is the central generator, which satisfies

$$\left[\mathbf{k}, \mathbf{T}_m^A\right] = 0, \quad \forall \mathbf{T}_m^A. \tag{34}$$

B. The 2-Sphere and 3-Sphere Algebras

The most immediate generalization of the loop group algebra (or Kac-Moody algebra) is the algebra corresponding to the Lie group $Map(M;G)$ of all smooth mappings from a compact manifold M, $\dim M > 1$, onto a compact simple Lie group G (with the group law of pointwise multiplication in G). The set of smooth mappings from a n-dimensional compact manifold M onto G will form an infinite-dimensional Lie group under pointwise multiplication.

We denote by $Map(S^n;G)$ the group of smooth mapping $\gamma : S^n \longmapsto G, \quad x \longmapsto \gamma(x)$ from the n-sphere to G; and by $Map(S^n;g)$ the corresponding infinite-dimensional Lie algebra. To see how the structure $Map(S^n;g)$ is derived for the cases of 2-Sphere and 3-Sphere algebras, let us consider a general element of the connected component of the group $Map(S^n;G)$ with $n = 2, 3$, which consists of elements

$$\gamma(x) = \exp\left(-i\mathbf{T}^A\theta_A(x)\right). \tag{35}$$

Near the identity they have the form

$$\gamma(x) \approx 1 - i\mathbf{T}^A\theta_A(x), \tag{36}$$

where the functions $\theta_A(x)$ are a set of $\dim(g)$ functions defined on S^n. These functions can be expanded in terms of a complete set or basis $\{F_I\}$ of orthogonal functions on S^n.

For functions on the 2-Sphere S^2 a basis is provided by the spherical harmonics $Y_{lm}(z,\varphi)$ ($z = \cos\theta$). For the 3-Sphere case there is also a complete set of orthogonal functions provided by the Wigner D-functions $D^j_{mm'}(\alpha,\beta,\gamma)$, where j is a non-negative integer or half-odd-integer and m, m' separately have the same spectrum in the range from $-j$ to j, changing in steps of one.

Expanding the functions $\theta_A(x)$ in the basis $\{F_I\}$

$$\theta_A(x) = \sum_I \theta^I_A F_I(x), \tag{37}$$

we can write

$$\gamma(x) \approx 1 - i\sum_I \theta^I_A \mathbf{T}^A F_I(x) = 1 - i\sum_I \theta^I_A \mathbf{T}^A_I, \tag{38}$$

where, for $n = 2$, $I \equiv L$ and $F_L = Y_L(z,\varphi)$, with L denoting the ordered pair (l,m); and for $n = 3$, $I \equiv J$ and $F_J = D_J(\alpha,\beta,\gamma)$, with J standing for the triple (j,m,m'). Identifying θ^I_A as the infinitesimal parameters of $Map(S^n, G)$ we see that $\mathbf{T}_I^A \equiv \mathbf{T}^A F_I$ are the generators of the algebra $Map(S^n, g)$. This algebra can be written as the product $g \otimes C^\infty(S^n)$, where $C^\infty(S^n)$ is the set of smooth functions on the n-dimensional sphere, S^n. The commutator is specified by

$$\left[\mathbf{T}^A_{I_1}, \mathbf{T}^B_{I_2}\right] = \left[\mathbf{T}^A F_{I_1}, \mathbf{T}^B F_{I_2}\right] = \left[\mathbf{T}^A, \mathbf{T}^B\right] \otimes F_{I_1} F_{I_2}, \tag{39}$$

where $F_{I_1}, F_{I_2} \in C^\infty(S^n)$. Is interesting to note that when the manifold is S^1, $Map(S^n, g)$ becomes the loop algebra.

To derive the expression for the commutation relations of the Lie algebra $Map(S^2, g)$ in the basis $\left\{\mathbf{T}^A_L\right\}$, let us first consider the direct product of two spherical harmonics of the same arguments. They may be expanded in series as

$$Y_{l_1 m_1} Y_{l_2 m_2} = \sum_l c(L_1, L_2, L) Y_{lm}, \tag{40}$$

where

$$c(L_1, L_2, L) = \langle l_1 m_1, l_2 m_2 \mid lm\rangle \langle l_1 0, l_2 0 \mid l0\rangle \left[\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}\right]^{1/2}. \tag{41}$$

Thus, the commutation relations for the generators $\mathbf{T}^A_L$ may now be obtained

$$\left[\mathbf{T}^A_{L_1}, \mathbf{T}^B_{L_2}\right] = \left[\mathbf{T}^A, \mathbf{T}^B\right] \otimes Y_{L_1} Y_{L_2} = \left(if^{ABC}\mathbf{T}^C\right) c(L_1, L_2, L) Y_L, \tag{42}$$

$$\left[\mathbf{T}^A_{L_1}, \mathbf{T}^B_{L_2}\right] = if^{ABC} c(L_1, L_2, L) \mathbf{T}^C Y_L, \tag{43}$$

$$\left[\mathbf{T}^{A}_{L_1},\mathbf{T}^{B}_{L_2}\right]=if^{ABC}c(L_1,L_2,L)\mathbf{T}^{C}_{L}, \tag{44}$$

where there is a summation over the dummy index L, that is, over $l\geq|m|$ since m has the fixed value m_1+m_2.

For the case of the group $Map(S^3,G)$ we have that $F_J\equiv D^{j}_{mm'}(\alpha,\beta,\gamma)$ providing a complete basis of functions on S^3, which may be expanded in series as

$$D^{j_1}_{m_1m_1'}D^{j_2}_{m_2m_2'}=\sum_{j}c(J_1,J_2,J)D^{j}_{mm'}, \tag{45}$$

where

$$c(J_1,J_2,J)=\langle j_1m_1,j_2m_2\mid jm\rangle\langle j_1m_1',j_2m_2'\mid jm'\rangle, \tag{46}$$

and the commutation relations for the generators $\mathbf{T}^{A}_{J}$ are given by

$$\left[\mathbf{T}^{A}_{J_1},\mathbf{T}^{B}_{J_2}\right]=if^{ABC}c(J_1,J_2,J)\mathbf{T}^{C}_{J}. \tag{47}$$

This algebra admits a nontrivial central extension, which is of the form

$$\left[\mathbf{T}^{A}_{I_1},\mathbf{T}^{B}_{I_2}\right]=if^{ABC}c(I_1,I_2,I)\mathbf{T}^{C}_{I}+d^{ab}_{I_1I_2(a)}\mathbf{K}^{a}, \tag{48}$$

where $\mathbf{K}^a$ is the central generators, which satisfies

$$\left[\mathbf{K}^{a},\mathbf{K}^{b}\right]=0;\quad\left[\mathbf{K}^{a},\mathbf{T}^{A}_{I}\right]=0\ , \tag{49}$$

where a is an index that labels the independent central elements.

IV. INFINITE-DIMENSIONAL LIE ALGEBRAS MAP($\mathbf{S}^n,g$) AS A S-EXPANDED LIE ALGEBRA

A. Loop Algebra as a S-expanded Lie algebra

If, in the S-expansion procedure, the finite semigroup is generalized to the case of an infinite semigroup, then the S-expanded algebra will be an *infinite-dimensional algebra*. One can this see from the fact that $\mathbf{T}_{(A,\alpha)}=\lambda_A\mathbf{T}_{\alpha}$ constitutes a base for the S-expanded algebra $\mathcal{G}$ and from the fact that α takes now the values in an infinite set.

A simple example is the case when the semigroup S becomes the infinite set of the integer numbers $\mathbb{Z}$ under the addition. The semigroup elements can be represented by the following

subset of the complex numbers $S = \{z^n = \exp(in\varphi)\}_{n=-\infty}^{\infty}$ for an arbitrary real parameter $\varphi \in [0, 2\pi]$ and n an integer. The semigroup composition law is given by

$$z^n z^m = z^{n+m} = \delta_l^{m+n} z^l. \tag{50}$$

The S-expansion procedure over the group manifold described in the section II has an interesting interpretation. According to the theorem 4, the relation between the coordinates of the original semigroup $\{\theta^A\}$ and the coordinates of the S-expanded semigroup $\{\theta^{(A,n)}\}$ takes the form

$$\theta^A = \sum_{n=-\infty}^{+\infty} \theta^{(A,n)} z^n. \tag{51}$$

This expression is equivalent to the Laurent expansion of the group coordinates defined over S^1 as in eq.(22) which leads to the loop algebra. This equivalence is due to the fact that the Fourier modes $z^n = \exp(in\varphi)$ obey the product law (50) of the semigroup $\mathbb{Z}$. In a few words, the S-expansion over the group manifold with the semigroup $\mathbb{Z}$ is equivalent to the "compactification" process over S^1 which leads to the definition of the loop algebra.

This means that if the pair $(g, [,])$ is a Lie algebra whose finite-dimensional vector space has a basis given by $\{\mathbf{T}_A\}_{A=1}^{\dim g}$, which satisfies the composition law $g \times g \longrightarrow g$, $(\mathbf{T}_A, \mathbf{T}_B) \longrightarrow [\mathbf{T}_A, \mathbf{T}_B] = if_{ABC}\mathbf{T}_C$, then the pair $(\mathcal{G}, [,])$ is a Lie algebra whose vector space has a basis given by $\{\mathbf{T}_{(A,n)}\}$, which satisfies a composition law $\mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$, defined by

$$\left[\mathbf{T}_{(A,m)}, \mathbf{T}_{(B,n)}\right]_S = z^m z^n \left[\mathbf{T}_A, \mathbf{T}_B\right] = z^m z^n \left(if_{ABC}\mathbf{T}_C\right), \tag{52}$$

where $\mathbf{T}_{(A,m)} = z^m \mathbf{T}_A$ is a basis of $\mathcal{G}$. Since $z^m z^n = z^{m+n} = \delta_l^{m+n} z^l$ we can write

$$\left[\mathbf{T}_{(A,m)}, \mathbf{T}_{(B,n)}\right]_S = if_{ABC}\delta_l^{m+n} z^l \mathbf{T}_C = if_{ABC}\delta_l^{m+n}\mathbf{T}_{(C,l)}, \tag{53}$$

$$\left[\mathbf{T}_{(A,m)}, \mathbf{T}_{(B,n)}\right]_S = if_{ABC}\mathbf{T}_{(C,m+n)}. \tag{54}$$

Comparing with eq.(31), we see that the above algebra obtained by an S-expansion with semigroup $\mathbb{Z}$ can be identified with the loop algebra.

B. The 2-Sphere and 3-Sphere algebras as S-expanded Lie algebras

In the case that the finite semigroup is generalized to a complete set or basis $\{F_I\}$ of orthogonal functions on S^n, we have that for the $n = 2$ case $S = \{F_I = Y_I(\theta, \varphi)\}$ and for the

$n = 3$ case $S = \{F_I = D_I(\alpha\beta\gamma)\}$; so that the relation between the coordinates of the original semigroup $\{\theta^A\}$ and the coordinates of the S-expanded semigroup $\{\theta^{(A,n)}\}$ according to the theorem 4 takes the form

$$\theta^A = \sum_{n=-\infty}^{+\infty} \theta^{(A,I)} F_I. \tag{55}$$

As in the loop algebra case, we can interpret this expression as a Fourier expansion in the base defined over the corresponding compact manifold (in this case S^n), where the semigroup elements are identified with the elements of the corresponding base.

The generators of the S-expanded algebra are given by $\mathbf{T}_{(A,I)} = \mathbf{T}_A F_I$ which satisfy a composition law defined by

$$\left[\mathbf{T}_{(A,I_1)}, \mathbf{T}_{(B,I_2)}\right] = \left[\mathbf{T}_A F_{I_1}, \mathbf{T}_B F_{I_2}\right] = \left[\mathbf{T}_A, \mathbf{T}_B\right] \otimes F_{I_1} F_{I_2}. \tag{56}$$

This expression defines the n-sphere Lie algebra.

V. LOOP ALGEBRA AND YANG-MILLS THEORY

In this section we consider the construction of actions invariant under the loop algebra.

A. Invariant tensors for loop algebras

In ref. [6], was show that the S-expansion procedure permits to obtaining invariant tensors for expanded Lie algebras from invariant tensors of the original Lie algebra. This allows constructing gauge theories such as Yang-Mills or Chern-Simons theories invariant under S-expanded Lie algebras.

Following ref.[6] it is direct to see that if $\langle \mathbf{T}_{A_1} \cdots \mathbf{T}_{A_N} \rangle$ is an invariant tensor for a Lie algebra g, then an invariant tensor for the loop algebra is given by:

$$\left\langle \mathbf{T}_{A_1}^{n_1} \cdots \mathbf{T}_{A_N}^{n_N} \right\rangle = \sum_{m=-\infty}^{+\infty} \alpha^{(m)} \delta_m^{n_1+n_2+\cdots+n_N} \left\langle \mathbf{T}_{A_1} \cdots \mathbf{T}_{A_N} \right\rangle, \tag{57}$$

where $\alpha^{(m)}$ are arbitrary constants.

B. Yang-Mills theory for the loop algebra of $SU\left(N\right)$

We consider the Lie algebra $g=SU\left(N\right)$, whose Killing metric is given by

$$\left\langle \mathbf{T}_{A}\mathbf{T}_{B}\right\rangle =\frac{1}{2}\delta_{AB}. \tag{58}$$

This invariant tensor permits constructing a Yang-Mills action given by

$$S_{YM}=-\frac{1}{4}\int F_{\mu\nu}^{A}F^{\mu\nu B}\left\langle \mathbf{T}_{A}\mathbf{T}_{B}\right\rangle . \tag{59}$$

If we consider the S-expansion of the Lie algebra $SU\left(N\right)$, we find that the associated Loop algebra is

$$\left[\mathbf{T}_{A}^{n},\mathbf{T}_{B}^{m}\right]=if_{ABC}\mathbf{T}_{C}^{n+m}, \tag{60}$$

where f_{ABC} are the structure constants of $SU\left(N\right)$. The corresponding invariant tensor for the Loop algebra can be directly obtained:

$$\left\langle \mathbf{T}_{A}^{n}\mathbf{T}_{B}^{m}\right\rangle =\frac{1}{2}\sum_{r=-\infty}^{+\infty}\alpha^{(r)}\delta_{r}^{n+m}\delta_{AB}, \tag{61}$$

where $\alpha^{(r)}$ are arbitrary constants. This invariant tensor permits constructing Yang-Mills actions for Loop algebras. In fact, the Yang-Mills action for the loop algebra obtained from the Lie algebra $SU\left(N\right)$ is

$$S=-\frac{1}{4}\int d^{d}x\left\langle \mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}\right\rangle , \tag{62}$$

where $\left\langle \cdots\right\rangle$ corresponds to the invariant tensor defined in (61).

The loop-algebra valued curvature $\mathbf{F}_{\mu\nu}=\sum\limits_{n=-\infty}^{+\infty}F_{\mu\nu(n)}^{A}\mathbf{T}_{A}^{n}$ is given by

$$\mathbf{F}_{\mu\nu}=\partial_{\mu}\mathbf{A}_{\nu}-\partial_{\nu}\mathbf{A}_{\mu}+i\left[\mathbf{A}_{\mu},\mathbf{A}_{\nu}\right], \tag{63}$$

where $\mathbf{A}_{\mu}=\sum\limits_{n=-\infty}^{+\infty}A_{\mu(n)}^{B}\mathbf{T}_{B}^{n}$ is the gauge potential. Thus the components of the curvature are given by

$$F_{\mu\nu(n)}^{B}=\partial_{\mu}A_{\nu(n)}^{B}-\partial_{\nu}A_{\mu(n)}^{B}-f_{CDB}\sum_{k=-\infty}^{+\infty}A_{\mu(n-k)}^{C}A_{\nu(k)}^{D}. \tag{64}$$

The corresponding gauge transformations are

$$\delta\mathbf{A}_{\mu}=\partial_{\mu}\mathbf{\Lambda}+i\left[\mathbf{A}_{\mu},\mathbf{\Lambda}\right], \tag{65}$$

where $\mathbf{\Lambda}$ is the gauge parameter. In terms of its components we can write

$$\delta A_{\mu(n)}^{B}=\partial_{\mu}\Lambda_{(n)}^{B}-f_{CDB}\sum_{k=-\infty}^{+\infty}A_{\mu(n-k)}^{C}\Lambda_{(k)}^{D}. \tag{66}$$

Introducing the invariant tensor (61) into the action (62), we obtain the Yang-Mills action for a loop algebra (60)

$$S=-\frac{1}{8}\sum_{r,n,=-\infty}^{+\infty}\alpha^{(r)}\int d^{d}xF_{\mu\nu(n)}^{A}F_{(r-n)}^{A\mu\nu}. \tag{67}$$

The coefficients $\alpha^{(r)}$ are arbitrary. The action will be invariant independently of the values that the above mentioned coefficients acquire.

VI. COMMENTS

We have generalized the S-expansion method of Lie algebras to the group manifold, where the expansion occurs at the level of the group coordinates as is stated in the theorem 4. The consistency with the dual formulation of the S-expansion procedure has also been checked. This generalization has been useful to show that the Lie algebras of smooth mappings of some manifold M onto a finite-dimensional Lie algebra , such as the so called Loop algebras, can be interpreted as particular kinds of S-expanded Lie algebras.

The main results of this paper are: the interpretation of the infinite-dimensional Lie-algebra $Map(S^{n},g)$ as a particular kind of S-expanded Lie algebras, where the notion of the S-expansion is generalized to include the group manifold. The obtaining of invariant tensors for loop algebras (considered as expanded algebras) allow constructing gauge theories such as Yang-Mills or Chern-Simons theories invariant under loop-like algebras.

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